setting $r^2 = \eta$, one obtains

$$s = -\kappa \int_0^\infty e^{-\kappa\xi}(\xi + 1)[\eta + (\xi + 1)^2]^{-\frac{1}{2}} \Big|_{\eta=0}^{\infty} d\xi = 1. \tag{24}$$

Thus, the reactant is consumed completely at the surface, for any arbitrary value of κ. This result is, of course, meaningless for $\kappa = 0$.

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FURTHER RESULTS FROM USE OF A TRANSCENDENTAL PROFILE FUNCTION IN CONDUCTION AND CONVECTION

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NOMENCLATURE

A,	coefficient;
a,	profile parameter, function of n;
b,	profile parameter, function of n;
C,	coefficient;
c,	specific heat;
$-Ei(-u)$,	exponential integral;
$E_p(u)$,	general exponential integral;
f,	coefficient;
g,	variable;
G_n,	function related to exponential;
$i^m \operatorname{erfc} \eta$,	mth repeated integral of the error function;
J,	coefficient;
K,	coefficient;
k,	thermal conductivity; also, running index;
n,	exponent;
p,	general function; running index;
q_i,	ith generalized coordinate; q_1, surface temperature; q_2, penetration depth;
T,	arbitrary constant temperature;
t,	time;
u,	$\exp(a + b\eta)$;
u_0,	$\exp a$;
x, y, z,	space coordinates;
pro,	dimensionless temperature profile approximation.

Greek symbols

α,	thermal diffusivity;
β,	profile parameter;
$\Gamma(1 + n)$,	gamma function;
ε,	an arbitrary small number; also, profile parameter;
η,	general variable;

θ,	temperature;
ρ,	density;
τ,	dimensionless time;
ϕ,	normalized function;
χ,	dimensionless penetration depth;
ψ,	dimensionless surface temperature, $x = 0$.

1. INTRODUCTION

CONSIDER some continuous, finite, non-negative, monotonic function $p(\eta)$ in the range $0 \leqslant \eta \leqslant \infty$. Let it be normalized with respect to its range, i.e. put $\phi(\eta) = [p - \min(p)]/[\max(p) - \min(p)]$, so that $0 \leqslant \phi \leqslant 1$. A simple transcendental approximation for ϕ is pro (η), where

$$\text{pro}(\eta) = \exp[-\exp(a + b\eta)]/\exp[-\exp a], \qquad (1)$$

provided that

$$\exp[-\phi'(0)\eta] - \phi < 0 \qquad \text{for all} \quad \eta > 0.$$

The conditions satisfied by this function (or its complement) are typical of those satisfied by temperature distributions in transient conduction or in convection, and by boundary-layer velocity profiles. In two previous papers [1, 2] the author has applied the approximation, equation (1), to two convection problems with favorable results. In this paper the approximation is applied to three further problems, two involving transient conduction and the third, convection.

2. FIRST PROBLEM: TRANSIENT ONE-DIMENSIONAL CONDUCTION WITH A NON-LINEAR BOUNDARY CONDITION

This is basically the same problem investigated previously [3] but with use then of a low-order polynomial approximation. If the x-axis is normal to the surface of the semi-infinite slab being considered, it is convenient to write

$$\eta = x/q_2, \qquad u = \exp(a + b\eta) \qquad \text{and} \qquad u_0 = \exp a$$

so that the profile assumed for the temperature distribution is

$$\theta(\eta) = q_1 \exp(-u)/\exp(-u_0). \qquad (2)$$

It is also assumed that the heat flux boundary condition F at $x = 0$ belongs to the class of functions

$$F = f\theta^n = fq_1^n. \qquad (3)$$

The variables can be transformed to $\psi = q_1/T$, $\chi = q_2 f T^{n-1}/k$, and $\tau = tf^2T^{2(n-1)}/kc\rho$, as in [3], so that the equations corresponding to equations (10) and (11) of [3], determined from application of Biot's variational principle [4, 5], are

$$J\dot{\psi} + (\tfrac{3}{2})A\dot{\psi}\chi\dot{\chi} + A\dot{\psi}\chi^2 = 0 \qquad (4)$$

and (from the heat flux boundary condition)

$$\dot{\psi}\chi + \psi\dot{\chi} = K\psi^n. \qquad (5)$$

The coefficients in these equations differ from those in [3] because of the different assumed profile. These equations can be solved for the cases $n = 0$ (for which there is also an exact solution) and $0 < n < 1$. The solutions are similar to those in [3] except for numerical values of coefficients. (In [3] these sets of solutions were incorrectly indicated to be asymptotic solutions instead of complete solutions.) Details of the method of solution, which involves determination of the coefficients in terms of exponential integrals [6, 7], can be found elsewhere [8].

Constant flux solution. The values found for the solutions were

$$\psi = 1{\cdot}130\,\tau^{\frac{1}{2}}, \qquad \chi = 3{\cdot}18\,\tau^{\frac{1}{2}} \qquad (6)$$

which can be compared with the exact solution

$$\psi = 1{\cdot}1284\,\tau^{\frac{1}{2}}.$$

The coefficient $\psi/\tau^{\frac{1}{2}}$ here differs from the exact solution by about 0·2 per cent and from the parabolic approximation by about 2·4 per cent. A comparison has been made in Fig. 1. The abscissa, the slab depth, has been adjusted so

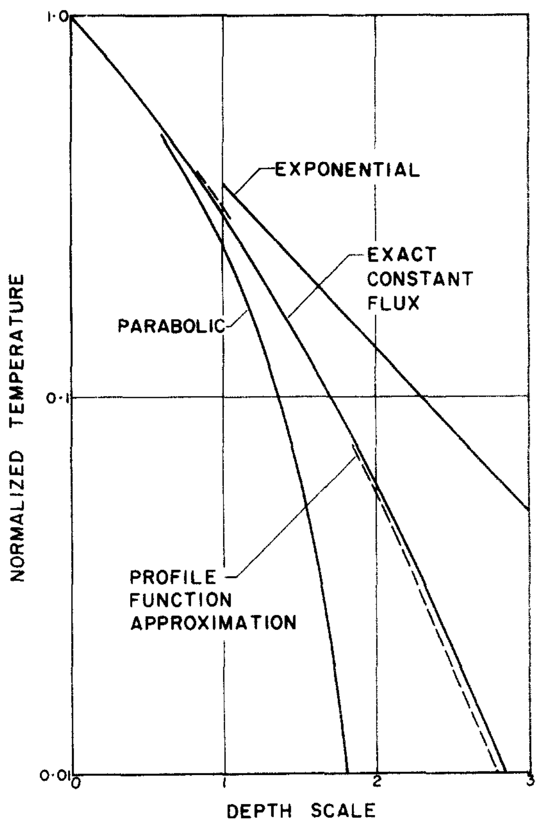

FIG. 1. Comparison of the exact temperature profile for the constant flux solution with various approximations. The approximation function used here (dashed curve) gives a very close approximation to the exact profile.

that all profiles shown have a slope of unity at the body surface. This figure demonstrates well that the profile function used here is a close approximation.

Solutions for $n > 0$. A set of solutions was computed for a range of n where the computations did not become inordinately long. In Table 1 values of a and b are provided, and in Table 2 values of the surface temperature coefficients $\psi/\tau^{\frac{1}{2}(1-n)}$ (parabolic profile), $\psi/\tau^{\frac{1}{2}(1-n)}$ (transcendental profile) and penetration depth coefficient (transcendental profile) are listed, together with the percentage differences of the surface temperature coefficients. As n increases towards unity, a tends to a large positive value, which means that the profile approaches a simple negative exponential in shape, i.e. pro $\eta \to \exp[-(b \exp a)\eta]$. This is expected.

Table 1. Solutions for $n > 0$

n	a	b
0·1	1·13287	0·90959
0·2	1·21696	0·86023
0·3	1·31709	0·80370
0·4	1·43553	0·73998
0·5	1·57704	0·66855
0·6	1·75592	0·58528
0·7	1·98445	0·49042
0·73	2·06623	0·45951

$b = \ln(1 - e^{-a} \ln \varepsilon)$;
$\varepsilon = 0{\cdot}01$ used to represent typical penetration depth.

Table 2

n	Surface temperature coefficients $\psi/\tau^{\frac{1}{2}(1-n)}$ parabolic profile	Surface temperature coefficients $\psi/\tau^{\frac{1}{2}(1-n)}$ transcendental profile	Penetration depth $\chi/\tau^{\frac{1}{2}}$ transcendental profile	Difference in surface temperature coefficients (per cent)
0·1	1·1347	1·1053	3·09136	2·7
0·2	1·0996	1·0658	3·05915	3·2
0·3	1·0453	1·0064	3·01486	3·9
0·4	0·96170	0·91769	2·96901	4·8
0·5	0·83340	0·78593	2·92637	6·0
0·6	0·64006	0·59309	2·76203	7·9
0·7	0·37095	0·33493	2·57070	10·7
0·73	0·28085	0·25068	2·49858	12·0

3. SECOND PROBLEM: APPROXIMATION OF THE REPEATED INTEGRALS OF THE ERROR FUNCTION

When conduction occurs in a semi-infinite homogeneous solid initially at a uniform temperature with the boundary condition

$$\theta(0, t) = pt^{n/2}$$

the solution is

$$\theta(x, t) = p\Gamma(n/2 + 1)(4t)^{n/2}\, i^n \operatorname{erfc}\left(\frac{x}{2\sqrt{(\alpha t)}}\right). \tag{7}$$

Carslaw and Jaeger [9] remark that if a particular $\theta(0, t)$ can be approximated by a series

$$\theta(0, t) = \Sigma p_n t^{n/2} \tag{8}$$

then the solution is simply the corresponding sum of terms as given in equation (7). This method of solution is used less frequently than it might be largely because of the trouble of handling the repeated integrals of the error function. To use equation (1) as an approximation, values for (a_n, b_n) to represent $i^n \operatorname{erfc}(g)/i^n \operatorname{erfc}(0)$ must be found. These could be sought in a least-squares mode of approximation, but this is a major computational process. Here, the values for (a_n, b_n) are sought which satisfy the conditions that the first derivatives are equal at $g = 0$ (i.e. at $\eta = 0$), and that the first integrals of the normalized repeated integral and of the approximation are equal. The first condition leads to

$$b_n \exp a_n = \frac{2\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n-1}{2} + 1\right)} \tag{9}$$

and the second condition can be stated as

$$\int_0^\infty \frac{i^n \operatorname{erfc}(g)}{i^n \operatorname{erfc}(0)}\, dg = \int_0^\infty \operatorname{pro}(g)\, dg. \tag{10}$$

The two sides of this equation can be reduced separately, such that the equation can be written

$$\frac{\Gamma\left(\frac{n}{2} + 1\right)}{2\Gamma\left(\frac{n+1}{2} + 1\right)} = \frac{e^{\exp(a_n)}}{b_n} E_1(\exp a_n) \tag{11}$$

where E_1 is the exponential integral. By combining equations (9) and (11), b_n can be eliminated to give

$$\frac{\left\{\left(\Gamma\,\frac{n}{2}+1\right)\right\}^2}{\Gamma\left(\frac{n-1}{2}+1\right)\Gamma\left(\frac{n+1}{2}+1\right)} \equiv G_n$$
$$\equiv (\exp a_n)\, e^{\exp a_n} E_1(\exp a_n). \qquad (12)$$

The values of G_n range from $G_0 = 0{\cdot}63661977$ to $G_n \to 1{\cdot}0$ as $n \to \infty$. A set of values for a_n and b_n is listed in Table 3. The calculation of a_n for further values of n is particularly easy; the right-hand side of equation (12) has the form $x\,e^x E_1(x)$, which is a well-known function for large x.

The differences between the normalized nth repeated integrals and the corresponding approximations using the values for a_n in Table 3 were computed for $n = 1, 2, 3, 4, 5, 6, 10$. The absolute error was found to be rather small, with a maximum of about 0·0025 for $n = 1$ at $\eta = 0{\cdot}500$. The variation of error with η is illustrated in Fig. 2 for $n = 1, 5$ and 10. Approximate error extrema are listed in Table 3 for all n computed. The absolute errors found here should be completely negligible in applications to thermal or material diffusion. The approximation can also be used to generate good approximations for values of n for which no tabulations now exist.

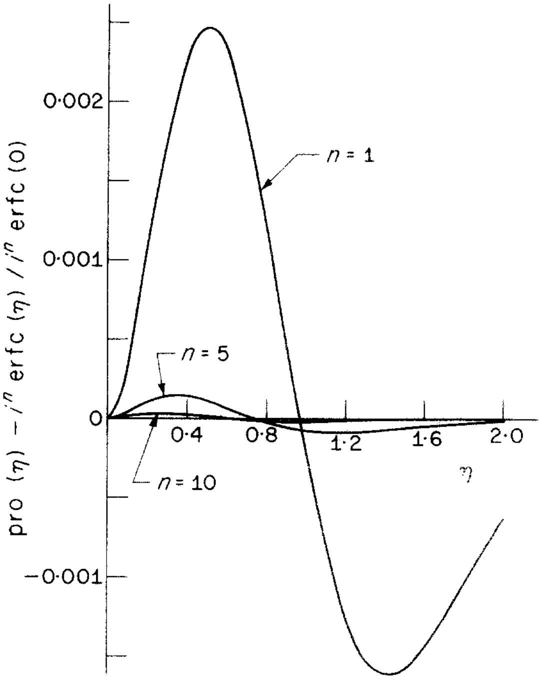

FIG. 2. Curves illustrating error of approximation of normalized repeated integral of the error function. The error decreases rapidly as the order of the integral increases.

4. THIRD PROBLEM: LAMINAR FORCED CONVECTION ANALYSIS

Hanson and Richardson [1] studied the class of wedge-flow boundary layers, approximating the velocity profiles with $1 - \text{pro}\,(\eta)$. The integral momentum and energy equations were used to provide sufficient means for evaluation of a and b for various wedge angles. The corresponding wall stress did not vary by more than 0·3 per cent from Hartree's solution for $m > -0{\cdot}03$. It might be expected on the basis of this experience that the profile function would give similar improvements in accuracy over polynomials when used to approximate temperature profiles in laminar boundary layers. However, this does not prove to be so, and an understanding of the reason for this (discussed below) gives insight into why the approximation can be prevented from achieving its potential in some problems.

Table 3

	Constants for approximation function				Extrema of error	
n	G_n	$\exp a_n$	a_n	b_n	1st extremum	2nd extremum
1	0·785398	2·983	1·093	0·5941	0·00247	−0·00161
2	0·848826	4·858	1·581	0·4644	0·00094	−0·00057
3	0·883573	6·784	1·915	0·3917	0·00046	−0·00026
4	0·905415	8·735	2·167	0·3446	0·00024	−0·00015
5	0·920388	10·700	2·370	0·3107	0·00014	−0·00009
6	0·931824	12·674	2·540	0·2848	0·00010	−0·00005
7	0·939563	14·654	2·685	0·2646	...	...
8	0·946066	16·639	2·812	0·2480	...	...
9	0·951308	18·624	2·924	0·2342	...	...
10	0·955622	20·612	3·026	0·2224	0·00003	−0·00001

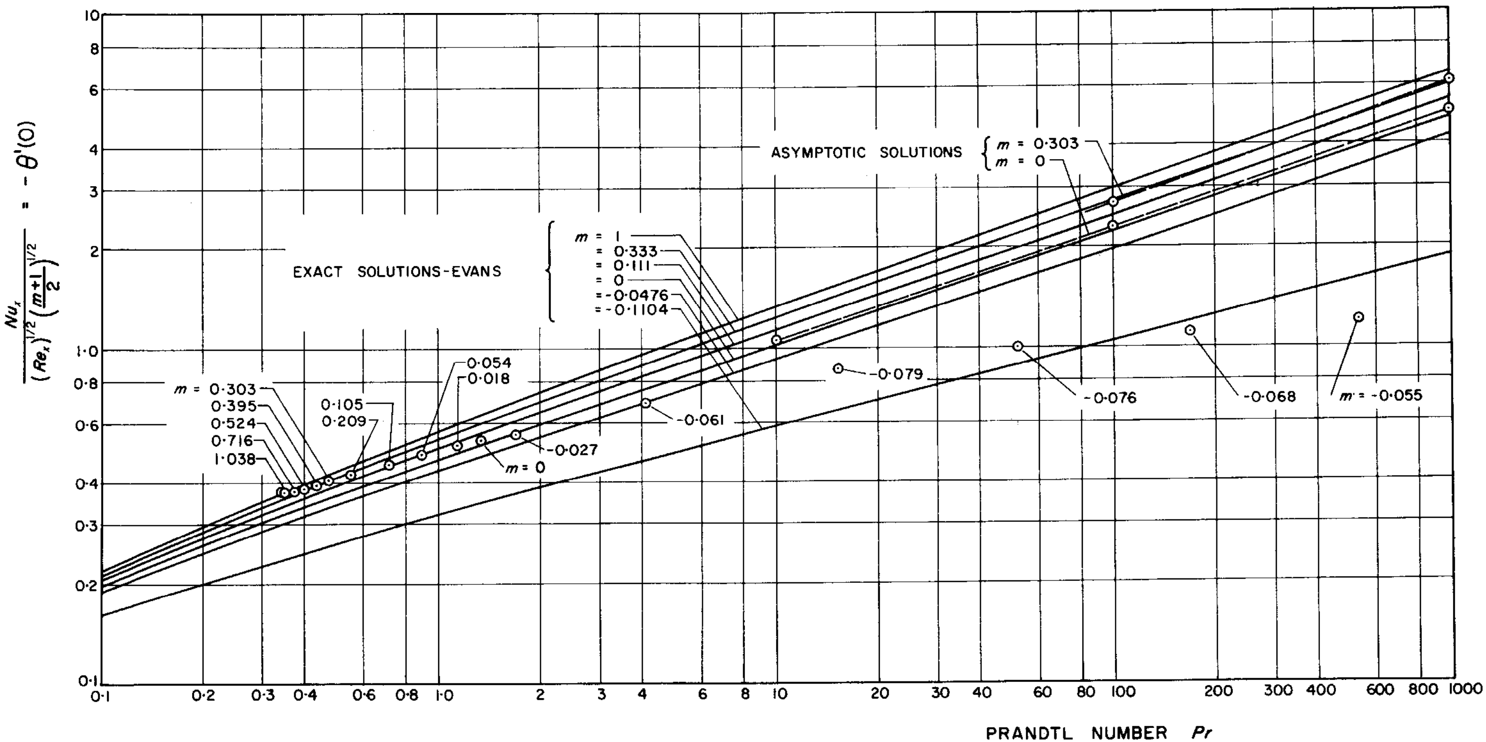

FIG. 3. Dimensionless heat transfer as a function of Prandtl number and wedge parameter, m. All points are for solution of the integral thermal energy equation. The points for which values of m are separately identified are for values of Prandtl number where $b = \beta$.

The approximate solution for the velocity profile can be used in the differential equation for thermal energy, and this can then be integrated subject to simple boundary conditions. Of the three successive integrations required the first can be done in closed form, but the remaining integrals are rather involved and numerical procedures become necessary because virtually no tabulations exist for repeated and for modified integrals of the exponential integral. This does not give the approximate velocity profile a significant advantage over use of a purely numerical solution for the velocity; but for heat-transfer calculations via the Meksyn–Merk method [10] the accuracy is good.

Alternatively, a profile approximation for the dimensionless temperature can be assumed as well as for the velocity. This was written

$$\theta(\eta) = \exp\left[-\exp(\varepsilon + \beta\eta)\right]/\exp\left[-\exp\varepsilon\right]. \tag{13}$$

Two conditions are required to determine $\varepsilon(m, Pr)$ and $\beta(m, Pr)$. The integral thermal energy equation is clearly one. However, it was not found possible to obtain a straightforward second integral equation without a double integration remaining; this is not convenient, as noted in the previous paragraph, so the compatibility condition at the wall

$$(\partial^2\theta/\partial y^2)_{y=0} = 0 \tag{14}$$

was used. This equation leads immediately to the result that either $\beta = 0$ (trivial), $\varepsilon = -\infty$, or $\varepsilon = 0$. Only the choice $\varepsilon = 0$ can be used; the compatibility condition has fixed the *shape* of the temperature profile. One of the integrals arising from the integral thermal energy equation can be written simply as an exponential integral, but the other has the form

$$\int_0^\infty \frac{\exp\left[-C(t^{\beta/b} + t)\right]}{t}\,\mathrm{d}t, \tag{15}$$

and this integral does not have a known solution except for $\beta = b$. (The case $\beta = 0$ is trivial.) With the restrictions to $\varepsilon = 0$, $\beta = b$, it is to be expected that for a particular value of m, the energy integral equation will be satisfied at only one Prandtl number. Values of Pr were computed, making use of Pagurova's tables [6], for the cases already considered in the wedge-flow velocity studies. The results are plotted in Fig. 3 with $-\theta'(0)$ against Pr in logarithmic coordinates; a selected portion of the very accurate results of Evans [11] is also displayed. The error in $\theta'(0)$ for each pair of m and Pr is about 3 per cent (ignoring the results for $m < -0{\cdot}03$). Thus the accuracy is not as good as for the velocity profiles.

Asymptotic solutions can be obtained by making suitable expansions of the velocity and temperature profiles; this eliminates the restriction $\beta = b$ considered previously, but retains the restriction $\varepsilon = 0$. When compared with the results of Evans, the error for both large and small Prandtl numbers was found to be about 3 per cent.

Thus the use of the compatibility condition fixes the shape of the profile, which is determined by the parameter ε, and thereby robs the approximation of one of its virtues: that the shape of the profile can normally vary to suit the situation.

5. CONCLUSIONS

The new results obtained for the first conduction problem are clearly of greater accuracy than those found previously [3]. The results derived for the second conduction problem provide new convenience in computation (or programming for computation) for problems involving the boundary conditions described adequately by equation (8).

These studies have been directed largely towards gaining an understanding of the usefulness of a transcendental approximation in obtaining a good representation of a profile shape in general and of the gradient at the origin in particular.

The transcendental approximation, equation (1), appears to offer good prospects of superior accuracy for problems where (i) the profiles are not expected to involve marked inflections, or to be close to the simple negative exponential (ii) the methods used do not require repeated integrations of the profile, and (iii) no constraints arise which fix the value of a before integral relations are utilized. Many problems satisfy these conditions.

The other favorable feature of using a transcendental approximation is that it may be able to give fairly uniform approximation of derivatives for a large range of η. While promising results were obtained in use [1, 2], there is insufficient experience available to offer conclusions about general utility.

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CONDUCTION IN NONGRAY RADIATING GASES

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NOMENCLATURE

A_i,	total band absorptance of the *i*th band;
A_{oi},	correlation parameter for the *i*th band;
B_i^2,	correlation parameter for the *i*th band;
C_{oi}^2,	correlation parameter for the *i*th band;
e_ω,	Planck's function;
$e_{\omega i}$,	Planck's function evaluated at the center of the *i*th band;
E_n,	exponential integral;
k,	thermal conductivity;
P,	pressure;
P_{ei},	equivalent broadening pressure for the *i*th band;
q_c,	conductive heat-transfer rate;
q_r,	radiative heat-transfer rate;
T,	temperature;
u_i, u_i^+, u_i^-,	dimensionless coordinates for the *i*th band; C_{oi}^2Py, $\frac{3}{2}C_{oi}^2P\delta(\xi+\eta)$ and $\frac{3}{2}C_{oi}^2P\delta(\xi-\eta)$ respectively;
x,	dummy variable for y;
y,	coordinate normal to plates.

Greek symbols

δ,	distance between plates;
ε,	wall emittance;
η,	dummy variable for ξ;
θ,	dimensionless temperature ratio, T/T_2;
κ_ω,	spectral absorption coefficient;
κ_P,	Planck mean coefficient;
κ_R,	Rosseland mean coefficient;
ξ,	dimensionless coordinate, y/δ;
ρ,	wall reflectance.

Subscripts

1,	lower wall;
2,	upper wall.

THE GRAY-GAS approximation has been extensively used to study the interaction of gaseous radiation with the other modes of heat transfer. However, there have been very few investigations, for example [1] and [2], which have dealt with the validity of the gray-gas model. This note is concerned with the application of a wide-band nongray gas model to conduction–radiation interaction in a static horizontal gas layer bounded by two walls. Gille and Goody [3] have also studied combined conduction and radiation in an ammonia gas layer; however, their investigation was primarily concerned with the stability of horizontal gas layers. The formulation of the radiative terms used here in the nongray analysis is similar to the analyses presented in [1] and [3]. The primary purpose of this note is to present representative results from a nongray